

# Supercurrent in superconducting graphene

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The problem of supercurrent in superconducting graphene is revisited and the supercurrent is calculated within the mean-field model employing the two-component wave functions on a honeycomb lattice with pairing between different valleys in the Brillouin zone. We show that the supercurrent within the linear approximation in the order-parameter-phase gradient is always finite even if the doping level is exactly zero.

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## I. INTRODUCTION

Recent exciting developments in transport experiments on graphene<sup>1</sup> have stimulated theoretical and experimental studies of possible superconductivity phenomena in this material. Experimentally, there are both hints towards intrinsic superconductivity<sup>2</sup> and observations of proximity-induced superconductivity in graphene layers<sup>3</sup>. Intrinsic superconductivity has been discussed theoretically in the frameworks of phonon and plasmon mediated mechanisms<sup>4,5</sup> whereas resonating valence bond and density wave lattice models were proposed in Refs. 6–8. It was shown within the BCS model<sup>7,9,10</sup> that the superconducting transition in the undoped graphene possesses a quantum critical point at a finite interaction strength below which the critical temperature vanishes. However, electrons in graphene may become unstable towards formation of Cooper pairs for any finite pairing interaction if doping shifts the Fermi level by an amount  $\mu$  away from the Dirac point<sup>4,7,8,10</sup>. The effect of fluctuations on the critical temperature of superconducting transition in graphene has been studied in Ref. 11. A number of unusual features of superconducting state have been predicted, which are closely related to the Dirac-like spectrum of normal state excitations. In particular, the unconventional normal electron dispersion has been shown to result in a nontrivial modification of Andreev reflection<sup>12</sup> and Andreev bound states in Josephson junctions<sup>13</sup> and vortex cores (see Ref. 14 and references therein).

Nevertheless, there still remains a controversy regarding the most fundamental property of superconducting graphene, i.e., the *supercurrent*, no matter what the mechanism, intrinsic or extrinsic, of the superconductivity is. In Ref. 7 the supercurrent has been calculated within the framework of the mean field model of superconducting graphene<sup>7,12</sup> that assumes the Cooper pairing between electrons belonging to the same sublattice in the configurational space. According to Ref. 7 the supercurrent calculated as a linear response to the phase gradient of the order parameter disappears in undoped

graphene (i.e., zero shift of the chemical potential,  $\mu = 0$ ) at zero temperature even if the order parameter  $\Delta$  itself is finite. However, a simpler model based on an effective Dirac type spectrum of normal electrons<sup>10</sup> demonstrates that the supercurrent is always finite as long as superconductivity exists,  $\Delta \neq 0$ . Though the surprising result<sup>7</sup> of “superconductivity without supercurrent” is an alarming indication by itself, the question may be raised, to which extent this difference between the supercurrents is model-dependent<sup>15</sup>, or, if not, what is then the correct behavior of the supercurrent in the low doping limit,  $\mu \rightarrow 0$ .

In the present paper we revisit this problem and calculate the supercurrent again using the two-component mean field model of superconductivity in graphene as formulated in Refs. 7,12. We show that the supercurrent in fact is *always finite*. Its value in the low doping limit  $\mu \ll |\Delta|$  is independent of whether the doping level is exactly zero or not, in contrast to the claim of Ref. 7. This statement qualitatively agrees with the conclusion drawn from the simple model suggested in Ref. 10.

The paper is organized as follows. In the next Section we outline the model of superconductivity in graphene as formulated in Refs. 7,12 and introduce the basic quantities relevant for further calculations. In Section III we calculate the supercurrent within the linear approximation in the order-parameter phase gradient for finite doping levels. The last Section IIIB deals with the case of low doping  $\mu \ll |\Delta|$ . Details of calculations are presented in Appendix A and Appendix B.

## II. BOGOLIUBOV–DE GENNES–DIRAC EQUATIONS

Transport properties of graphene associated with energies much smaller than the band width are conveniently described by equations of the Dirac type for two-component wave function whose two components are envelopes of the true wave functions for two sublattices in the configurational space, Fig. 1(a), near the so called Dirac points  $\mathbf{K}$  or  $\mathbf{K}'$  in the Brillouin zone of the reciprocal lattice, Fig. 1(b) (for more details see, for example,

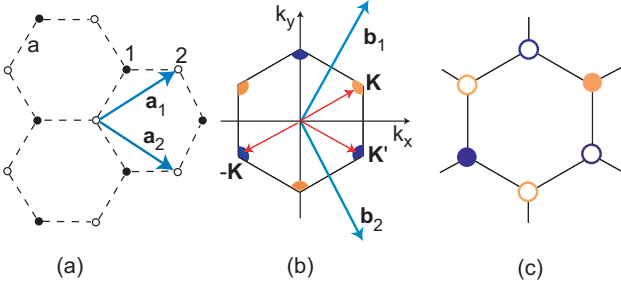


FIG. 1: (a) Unit cell with two sublattices 1 (black dots) and 2 (open dots), interatomic distance  $a$ , and the basis vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$ . (b) Brillouin zone with the reciprocal lattice vectors  $\mathbf{b}_1$  and  $\mathbf{b}_2$ .  $\mathbf{K}$  and  $\mathbf{K}'$  show the two non-equivalent Dirac corners (differently shaded sectors) of the Brillouin zone; other corners are obtained by shifting these two by integer linear combinations  $n_1\mathbf{b}_1 + n_2\mathbf{b}_2$ . (c) Dirac cone regions (circles) in the extended zone scheme. Filled circles belong to the same zone, while open circles are from other zones.

Refs. 12,16).

A hole-like excitation  $\Psi_{\mathbf{K}}^{(h)}$  in the valley associated with the point  $\mathbf{K}$  is the complex conjugated wave function of a particle-like excitation in the valley  $-\mathbf{K}$ , i.e.,  $\Psi_{\mathbf{K}}^{(h)} = \Psi_{-\mathbf{K}}^*$ . In what follows we denote particle-like states by  $u$  while hole-like states by  $v$ . The Bogoliubov-de Gennes equations have the form<sup>7,12</sup>

$$v_F \boldsymbol{\sigma} \cdot \left( -i\nabla - \frac{e}{c} \mathbf{A} \right) \hat{u}(\mathbf{r}) + \Delta \hat{v}(\mathbf{r}) = (\epsilon + \mu) \hat{u}(\mathbf{r}), \quad (1)$$

$$-v_F \boldsymbol{\sigma} \cdot \left( -i\nabla + \frac{e}{c} \mathbf{A} \right) \hat{v}(\mathbf{r}) + \Delta^* \hat{u}(\mathbf{r}) = (\epsilon - \mu) \hat{v}(\mathbf{r}). \quad (2)$$

The two-component wave functions are in a form of pseudo-spinors

$$\hat{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \hat{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \hat{u}^\dagger = (u_1^*, u_2^*), \quad \hat{v}^\dagger = (v_1^*, v_2^*),$$

where the two components are the wave functions of electrons and holes on two sublattices 1 and 2 in the honeycomb lattice, Fig. 1(a);  $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y)$  are Pauli matrices in the pseudo-spin space:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Equations for the valley at the point  $\mathbf{K}'$  can be obtained with the replacement  $u_1 \rightarrow u_2$ ,  $v_1 \rightarrow v_2$ .

Pairing of a particle  $u$  in the valley  $\mathbf{K}$  in the Brillouin zone occurs with a hole  $v$  at  $\mathbf{K}$ , i.e., with a particle in the valley  $-\mathbf{K}$ . Since the points  $-\mathbf{K}$  and  $\mathbf{K}'$  are equivalent,  $\mathbf{K} + \mathbf{K}' = \mathbf{b}_1 + \mathbf{b}_2$  where  $\mathbf{b}_1$  and  $\mathbf{b}_2$  are the vectors of the reciprocal lattice, Fig. 1(b), one may also say that pairing is between particles from the valleys  $\mathbf{K}$  and  $\mathbf{K}'$ . The model assumes that the order parameter is the same for both sublattices,

$$\Delta = -V \sum_{\mathbf{p}, \alpha} (1 - 2f_{\mathbf{p}, \alpha}) \hat{v}_{\mathbf{p}, \alpha}^\dagger \hat{u}_{\mathbf{p}, \alpha} \quad (3)$$

where  $\alpha$  labels four independent solutions of the Bogoliubov-de Gennes equations with the momentum  $\mathbf{p}$  (see below) and  $f_{\mathbf{p}, \alpha}$  is the Fermi occupation number in the state  $\mathbf{p}, \alpha$ . The sum runs over all states within the Brillouin zone. We do not concentrate here on the specific nature of the pairing interaction assuming that the pairing potential may be either due to some intrinsic mechanism or due to an interaction induced by a proximity to a usual superconductor.

The particle density is

$$N = 2 \sum_{\mathbf{p}, \alpha} [f_{\mathbf{p}, \alpha} \hat{u}_{\mathbf{p}, \alpha}^\dagger \hat{u}_{\mathbf{p}, \alpha} + (1 - f_{\mathbf{p}, \alpha}) \hat{v}_{\mathbf{p}, \alpha}^\dagger \hat{v}_{\mathbf{p}, \alpha}] .$$

Factor 2 accounts for the true spin of electrons. The statistical average of the current operator is

$$\mathbf{j} = 2ev_F \sum_{\mathbf{p}, \alpha} [\hat{u}_{\mathbf{p}, \alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p}, \alpha} f_{\mathbf{p}, \alpha} - \hat{v}_{\mathbf{p}, \alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{v}_{\mathbf{p}, \alpha} (1 - f_{\mathbf{p}, \alpha})] . \quad (4)$$

Sometimes the currents  $\mathbf{j}_e$  and  $\mathbf{j}_p$  are defined,

$$\mathbf{j}_e = -ev_F \sum_{\mathbf{p}, \alpha} [\hat{u}_{\mathbf{p}, \alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p}, \alpha} + \hat{v}_{\mathbf{p}, \alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{v}_{\mathbf{p}, \alpha}] (1 - 2f_{\mathbf{p}, \alpha}), \quad (5)$$

$$\mathbf{j}_p = ev_F \sum_{\mathbf{p}, \alpha} [\hat{u}_{\mathbf{p}, \alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p}, \alpha} - \hat{v}_{\mathbf{p}, \alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{v}_{\mathbf{p}, \alpha}] , \quad (6)$$

such that  $\mathbf{j} = \mathbf{j}_e + \mathbf{j}_p$ . The current  $\mathbf{j}_p$  is the quasiparticle flux, which vanishes in our spatially uniform case (see below). The current  $\mathbf{j}_e$  is the sum of currents in each state, which may be not conserved separately in some spatially inhomogeneous or non-equilibrium situations, but the total current, however, is conserved ( $\text{div} \mathbf{j}_e = 0$ ) taking into account the self-consistency equation<sup>17</sup>.

We will consider the case of zero magnetic field and look for the solution in the form of plane waves

$$\hat{u}_{\mathbf{p}} = \hat{u} e^{i(\mathbf{p} + \mathbf{k}/2) \cdot \mathbf{r}}, \quad \hat{v}_{\mathbf{p}} = \hat{v} e^{i(\mathbf{p} - \mathbf{k}/2) \cdot \mathbf{r}}, \quad (7)$$

assuming that the order parameter  $\Delta = |\Delta| e^{i\mathbf{k} \cdot \mathbf{r}}$  corresponds to a moving condensate of Cooper pairs. Equations (1) and (2) give

$$v_F \hat{\boldsymbol{\sigma}} \cdot (\mathbf{p} + \mathbf{k}/2) \hat{u} + \Delta \hat{v} = (E + \mu) \hat{u}, \quad (8)$$

$$-v_F \hat{\boldsymbol{\sigma}} \cdot (\mathbf{p} - \mathbf{k}/2) \hat{v} + \Delta^* \hat{u} = (E - \mu) \hat{v}. \quad (9)$$

### A. Ground state

Let us consider the ground state with zero current ( $\mathbf{k} = 0$ ). Equations (8), (9) define four linearly independent solutions. Let us introduce the spinors

$$\hat{a}_\uparrow = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x - ip_y}{p}} \\ \sqrt{\frac{p_x + ip_y}{p}} \end{pmatrix}, \quad \hat{a}_\downarrow = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x - ip_y}{p}} \\ -\sqrt{\frac{p_x + ip_y}{p}} \end{pmatrix}, \quad (10)$$

which satisfy

$$(\hat{\boldsymbol{\sigma}} \cdot \mathbf{p}) \hat{a}_{\uparrow, \downarrow} = \pm p \hat{a}_{\uparrow, \downarrow}. \quad (11)$$

The spinors  $\hat{a}_\uparrow$  and  $\hat{a}_\downarrow$  are eigenstates of excitations in the normal graphene. We also introduce vectors in the Nambu space,

$$\check{\psi} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}, \quad \check{\psi}^+ = (\hat{u}^\dagger, \hat{v}^\dagger).$$

Each component here is a pseudo-spinor. We find for the upper sign in Eq. (11)

$$E_{1,2}^{(0)} = \pm E_\uparrow, \quad E_\uparrow = \sqrt{(v_F p - \mu)^2 + |\Delta|^2}. \quad (12)$$

For  $\mathbf{k} = 0$  the order parameter is real  $\Delta = |\Delta|$ . Therefore,

$$\begin{pmatrix} \hat{u}_1^{(0)} \\ \hat{v}_1^{(0)} \end{pmatrix} = \begin{pmatrix} u_\uparrow \\ v_\uparrow \end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p} \cdot \mathbf{r}}, \quad \begin{pmatrix} \hat{u}_2^{(0)} \\ \hat{v}_2^{(0)} \end{pmatrix} = \begin{pmatrix} v_\uparrow \\ -u_\uparrow \end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p} \cdot \mathbf{r}}. \quad (13)$$

For the lower sign in Eq. (11) we have

$$E_{3,4}^{(0)} = \pm E_\downarrow, \quad E_\downarrow = \sqrt{(v_F p + \mu)^2 + |\Delta|^2}, \quad (14)$$

and

$$\begin{pmatrix} \hat{u}_3^{(0)} \\ \hat{v}_3^{(0)} \end{pmatrix} = \begin{pmatrix} u_\downarrow \\ v_\downarrow \end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p} \cdot \mathbf{r}}, \quad \begin{pmatrix} \hat{u}_4^{(0)} \\ \hat{v}_4^{(0)} \end{pmatrix} = \begin{pmatrix} v_\downarrow \\ -u_\downarrow \end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p} \cdot \mathbf{r}}. \quad (15)$$

Here

$$u_\uparrow = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{v_F p - \mu}{E_\uparrow}}, \quad v_\uparrow = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{v_F p - \mu}{E_\uparrow}}, \quad (16)$$

$$u_\downarrow = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{v_F p + \mu}{E_\downarrow}}, \quad v_\downarrow = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{v_F p + \mu}{E_\downarrow}}. \quad (17)$$

The different wave functions are orthogonal,  $\check{\psi}_\alpha^\dagger \check{\psi}_\beta = \delta_{\alpha\beta}$ . Equation (13) goes over into Eq. (15) under the transformation  $E \rightarrow -E$  and  $\mu \rightarrow -\mu$ . Using Eq. (10) one can check that

$$\hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\uparrow = -\hat{a}_\downarrow^\dagger \hat{\sigma} \hat{a}_\downarrow = \mathbf{p}/p, \quad (18)$$

$$\hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\downarrow = -\hat{a}_\downarrow^\dagger \hat{\sigma} \hat{a}_\uparrow = i[\mathbf{z}_0 \times \mathbf{p}]/p. \quad (19)$$

where  $\mathbf{z}_0$  is the unit vector in the  $z$  direction perpendicular to the graphene layer plane.

### III. CURRENT-CARRYING STATE

For the current-carrying state the solvability condition of the Bogoliubov–de Gennes equations (8), (9) takes the form

$$(E^2 - \mu^2)^2 - 2|\Delta|^2(E^2 - \mu^2) + |\Delta|^4 + 2|\Delta|^2 v_F^2 \mathbf{p}_+ \mathbf{p}_- - (E + \mu)^2 v_F^2 \mathbf{p}_-^2 - (E - \mu)^2 v_F^2 \mathbf{p}_+^2 + v_F^4 \mathbf{p}_+^2 \mathbf{p}_-^2 = 0, \quad (20)$$

where  $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$ .

Equation (20) cannot be solved analytically for nonzero  $\mathbf{k}$ , except for the zero doping limit  $\mu = 0$ . In

the latter case the solvability condition Eq. (20) becomes bi-quadratic and yields the energy spectrum

$$E_\pm^2 = |\Delta|^2 + v_F^2(p^2 + k^2/4) \pm \sqrt{|\Delta|^2 v_F^2 k^2 + v_F^4 (\mathbf{p} \cdot \mathbf{k})^2}. \quad (21)$$

In this limit, the Bogoliubov–de Gennes equations can also be solved analytically (see Appendix B 2). One sees that the energy, Eq. (21), for  $\mu = 0$  does not have the usual Doppler term proportional to the vector  $\mathbf{k}$ . This may lead to a confusion<sup>15</sup> when calculating the supercurrent.

#### A. Linear response

Let us consider the linear correction to the energy and to the wave functions due to superconducting momentum  $\mathbf{k}$  assuming  $v_F |\mathbf{k}| \ll \mu$ . We put  $E = E(0) + E'$  where  $E' \ll E(0)$  and  $E(0)$  is the energy of one of the states with  $\mathbf{k} = 0$  determined by Eqs. (12), (14). Within the linear approximation in  $\mathbf{k}$  we find from Eq. (20) for any finite  $\mu \neq 0$

$$E' = \pm v_F (\mathbf{p} \cdot \mathbf{k})/2p \equiv \pm E_D$$

for the upper (lower) sign in Eq. (11). Therefore, corrections to the energies are

$$E_{1,2}^{(1)} = -E_{3,4}^{(1)} = E_D. \quad (22)$$

The energy  $E_D = (d\xi_{\mathbf{p}}/dp)(\mathbf{k}/2)$  is the usual Doppler shift for the normal-state energy  $\xi_{\mathbf{p}} = v_F p$ . Equation (22) coincides with the result of Ref. 10 obtained in the linear approximation in  $\mathbf{k}$ . At the same time, it differs from the linear in  $\mathbf{k}$  term obtained from Eq. (21) for  $\mu = 0$ . This means that the undoped case  $\mu = 0$  requires a special consideration. This will be done later in Section III B (see also Appendix B).

First-order corrections to the wave functions can be found by expanding the total functions in terms of the zero-order functions  $\hat{u}_\beta^{(0)}, \hat{v}_\beta^{(0)}$  given by Eqs. (13), (15):

$$\check{\psi}_\alpha = \check{\psi}_\alpha^{(0)} + \sum_{\beta \neq \alpha} B_{\alpha\beta} \check{\psi}_\beta^{(0)}. \quad (23)$$

Inserting this into Eqs. (1), (2) we find

$$B_{\alpha\beta} = \frac{v_F \check{\psi}_\beta^{(0)+} (\hat{\sigma} \cdot \mathbf{k}) \check{\psi}_\alpha^{(0)}}{2(E_\alpha^{(0)} - E_\beta^{(0)})}.$$

One can check that  $B_{\beta\alpha} = -B_{\alpha\beta}^*$ . We find  $B_{12} = B_{21} = B_{34} = B_{43} = 0$  while

$$B_{13} = -B_{24} = -\frac{iv_F ([\mathbf{p} \times \mathbf{k}] \cdot \mathbf{z})}{2p} \frac{(u_\downarrow^* u_\uparrow + v_\downarrow^* v_\uparrow)}{E_\uparrow - E_\downarrow}, \quad (24)$$

$$B_{23} = B_{14} = \frac{iv_F ([\mathbf{p} \times \mathbf{k}] \cdot \mathbf{z})}{2p} \frac{(u_\downarrow^* v_\uparrow - v_\downarrow^* u_\uparrow)}{E_\uparrow + E_\downarrow}. \quad (25)$$

Therefore, the up-spin wave functions  $\hat{u}^{(1,2)}$  contain only corrections with the down-spin components  $\hat{u}^{(3,4)}$ , and vice versa. Expansion Eq. (23) yields also the corrections to the eigenenergies which coincide with Eq. (22).

Using Eq. (23) one can show that the quasiparticle current  $\mathbf{j}_p$  is zero. The supercurrent Eq. (4) takes the form of Eq. (5),  $\mathbf{j} = \mathbf{j}_e$ , which can be written as

$$\mathbf{j} = \int \frac{d^2p}{(2\pi)^2} [\mathbf{j}_{\mathbf{K}}(\mathbf{p}) + \mathbf{j}_{-\mathbf{K}}(\mathbf{p})] . \quad (26)$$

where

$$\mathbf{j}_{\mathbf{K}}(\mathbf{p}) = -ev_F \sum_{\alpha=1}^4 \hat{u}_{\mathbf{p},\alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p},\alpha} [1 - 2f_{\mathbf{p},\alpha}] , \quad (27)$$

$$\mathbf{j}_{-\mathbf{K}}(\mathbf{p}) = -ev_F \sum_{\alpha=1}^4 \hat{v}_{\mathbf{p},\alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{v}_{\mathbf{p},\alpha} [1 - 2f_{\mathbf{p},\alpha}] . \quad (28)$$

The term  $\mathbf{j}_{\mathbf{K}}(\mathbf{p})$  is the contribution from the valley  $\mathbf{K}$  in the Brillouin zone while  $\mathbf{j}_{-\mathbf{K}}(\mathbf{p})$  is the contribution from valley  $-\mathbf{K}$ . Therefore, Eq. (26) in fact collects contributions from the vicinity of the Dirac points at the opposite corners of the entire Brillouin zone [shaded sectors in Fig. 1(b) or (c)].

Using Eq. (23) we obtain from Eq. (5) in the linear approximation

$$\begin{aligned} \mathbf{j} = & -ev_F \sum_{\alpha,\mathbf{p}} \left[ \hat{u}_{\alpha}^{(0)\dagger} \hat{\boldsymbol{\sigma}} \hat{u}_{\alpha}^{(0)} + \hat{v}_{\alpha}^{(0)\dagger} \hat{\boldsymbol{\sigma}} \hat{v}_{\alpha}^{(0)} \right] \left[ 1 - 2f(E_{\alpha}^{(0)} + E_{\alpha}^{(1)}) \right] \\ & - 2ev_F \text{Re} \sum_{\alpha \neq \beta, \mathbf{p}} B_{\alpha\beta} \left[ \hat{u}_{\alpha}^{(0)\dagger} \hat{\boldsymbol{\sigma}} \hat{u}_{\beta}^{(0)} + \hat{v}_{\alpha}^{(0)\dagger} \hat{\boldsymbol{\sigma}} \hat{v}_{\beta}^{(0)} \right] \left[ 1 - 2f(E_{\alpha}^{(0)}) \right] . \end{aligned} \quad (29)$$

Equation (29) contains the terms which diverge for large  $v_F p \gg |\Delta|, T$  because of the contributions from the Fermi sea of the states with negative energies, which extend over the entire Brillouin zone including regions far from the Dirac point. This divergence is spurious and can be eliminated using two equivalent methods.

First, we note that the divergence of this kind is caused simply by the fact that the overall shift of the particle momentum in the Brillouin zone creates corrections to the wave functions which do not decay as functions of the momentum far from the Dirac points. Let us consider a change in the particle momentum  $\mathbf{p} \rightarrow \mathbf{p} + \delta\mathbf{p}$  everywhere in the Brillouin zone. It will lead to the shift  $\mathbf{p} \rightarrow \mathbf{p} + \delta\mathbf{p}$  in the functions  $u$  and, at the same time, to the shift  $\mathbf{p} \rightarrow \mathbf{p} - \delta\mathbf{p}$  in  $v$ , because the functions  $v$  are associated with the complex conjugated wave functions  $u$  taken at the point  $-\mathbf{K}$ . In this way, the wave functions used in Eq. (7) contain corrections associated with the overall shift  $\mathbf{p} \rightarrow \mathbf{p} + \mathbf{k}/2$  in the Brillouin zone. It is thus legitimate to simultaneously change the momentum under the integral in Eq. (26) or Eq. (29) back to its original value  $\mathbf{p}$ , i.e., to change the blind integration variable  $\mathbf{p} \rightarrow \mathbf{p} - \mathbf{k}/2$  in the first and  $\mathbf{p} \rightarrow \mathbf{p} + \mathbf{k}/2$  in second term. Excluding this momentum shift we thus remove the diverging part, which is not relevant to the supercurrent.

Within the linear approximation, it is sufficient to shift the momenta in the zero-order term which comes from

the first line of Eq. (29). The zero-order term yields

$$\begin{aligned} \mathbf{j}^{(0)} = & \int \frac{d^2p}{(2\pi)^2} [\mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p} - \mathbf{k}/2) + \mathbf{j}_{-\mathbf{K}}^{(0)}(\mathbf{p} + \mathbf{k}/2)] \\ = & \int \frac{d^2p}{(2\pi)^2} \left[ \mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p}) + \mathbf{j}_{-\mathbf{K}}^{(0)}(\mathbf{p}) - \left( \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{p}} \right) \mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p}) \right] . \end{aligned}$$

Here  $\mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p})$  and  $\mathbf{j}_{-\mathbf{K}}^{(0)}(\mathbf{p})$  are the currents Eqs. (27) and (28) within the zero-order approximation in  $\mathbf{k}$ , i.e., with the functions  $\hat{u}_{\alpha}^{(0)}$  and  $\hat{v}_{\alpha}^{(0)}$  and the energies  $E_{\alpha}^{(0)}$  of states Eqs. (12)–(17) without a current. For these states  $\mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p}) + \mathbf{j}_{-\mathbf{K}}^{(0)}(\mathbf{p}) = 0$ . As a result

$$\mathbf{j}^{(0)} = - \int \frac{d\phi}{(2\pi)^2} [(\mathbf{p} \cdot \mathbf{k}) \mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p})]_{p \gg \Delta, T} . \quad (30)$$

Here we transformed into the surface integral over a remote sphere in the momentum space. At these momenta and energies, the current in Eq. (30) does not contain any information on the superconducting properties of the material. This term compensates the divergence of the corrections in the second line of that equation.

Another way to remove the divergence in the second line of Eq. (29) would be to directly subtract from it the normal-state current which is identically zero for the wave functions specified by Eq. (7). Indeed, as we already mentioned, the diverging contributions to the current come from the regions far from the Dirac point. Since, at these quasiparticle momenta the energies

greatly exceed the scales relevant to the superconducting state, the corresponding contributions to the current coincide with those in the normal state. One can show that

this regularization procedure leads to the same result as Eq. (30). The details of calculations are given in Appendix A. The final expression for the current becomes

$$\mathbf{j} = ev_F^2 \mathbf{k} \int_0^\infty \frac{p dp}{2\pi} \left[ -\frac{1}{4T} \cosh^{-2} \frac{E_\uparrow}{2T} - \frac{1}{4T} \cosh^{-2} \frac{E_\downarrow}{2T} + \frac{|u_\uparrow^* u_\downarrow + v_\uparrow^* v_\downarrow|^2}{E_\uparrow - E_\downarrow} \left( \tanh \frac{E_\uparrow}{2T} - \tanh \frac{E_\downarrow}{2T} \right) + \frac{1}{v_F p} - \frac{|v_\uparrow^* u_\downarrow - u_\uparrow^* v_\downarrow|^2}{E_\uparrow + E_\downarrow} \left( \tanh \frac{E_\uparrow}{2T} + \tanh \frac{E_\downarrow}{2T} \right) \right]. \quad (31)$$

It is worthwhile to note that the problem of spurious divergent terms in the expression for the supercurrent is rather general. In particular, it was discussed (and resolved similarly) for the superfluid excitonic current in graphene bilayers<sup>18</sup>.

We evaluate Eq. (31) for low temperatures,  $T \ll |\Delta|$ . Since  $E_\uparrow, E_\downarrow > |\Delta|$ , the first line in Eq. (31) vanishes at  $T = 0$ . The supercurrent becomes

$$\mathbf{j} = \frac{e\mathbf{k}}{2\pi} \left[ \sqrt{\mu^2 + |\Delta|^2} + \frac{|\Delta|^2}{|\mu|} \ln \left( \frac{|\mu| + \sqrt{\mu^2 + |\Delta|^2}}{|\Delta|} \right) \right]. \quad (32)$$

For  $\mu \gg |\Delta|$  we have

$$\mathbf{j} = e|\mu|\mathbf{k}/2\pi. \quad (33)$$

For  $\mu \ll |\Delta|$  we find

$$\mathbf{j} = e|\Delta|\mathbf{k}/\pi. \quad (34)$$

This result formally holds within the linear approximation which assumes  $v_F k \ll \mu$ . Therefore, in Eq. (34) one would have to put  $k \rightarrow 0$  first and then assume  $\mu \ll |\Delta|$ . In the next section we demonstrate that Eq. (34) is in fact always valid provided  $|\mu| \ll |\Delta|$  and  $v_F k \ll |\Delta|$  irrespectively of the relation between  $v_F k$  and  $\mu$ .

## B. Low doping limit

Consider now the limit of small  $\mu$  when the zero order state is degenerate because  $E_1^{(0)} = E_3^{(0)} = E_0$  and  $E_2^{(0)} = E_4^{(0)} = -E_0$  where

$$E_0 = \sqrt{(v_F p)^2 + |\Delta|^2}. \quad (35)$$

In what follows we demonstrate that despite the absence of the Doppler term in its usual form, there still is a finite linear in  $\mathbf{k}$  supercurrent down to zero temperature (contrary to the result of Ref. 7).

The true wave functions satisfy

$$\check{H} \check{\psi}_\alpha = E_\alpha \check{\psi}_\alpha$$

where  $\check{H} = \check{H}^{(0)} + \check{H}^{(1)}$  and

$$\check{H}^{(0)} = \begin{pmatrix} v_F \hat{\boldsymbol{\sigma}} \cdot \mathbf{p} & |\Delta| \\ |\Delta| & -v_F \hat{\boldsymbol{\sigma}} \cdot \mathbf{p} \end{pmatrix}, \quad \check{H}^{(1)} = \begin{pmatrix} \frac{1}{2} v_F \hat{\boldsymbol{\sigma}} \cdot \mathbf{k} - \mu & 0 \\ 0 & \frac{1}{2} v_F \hat{\boldsymbol{\sigma}} \cdot \mathbf{k} + \mu \end{pmatrix}.$$

We assume  $v_F k, \mu \ll E_0$ .

Let us expand the true wave function into the zero order orthonormal wave functions  $\check{\psi}_\alpha^{(0)}$ ,

$$\check{\psi}_\alpha = \sum_\beta C_{\alpha\beta} \check{\psi}_\beta^{(0)}, \quad (36)$$

satisfying the zero-order equation

$$\check{H}^{(0)} \check{\psi}_\alpha^{(0)} = E_\alpha^{(0)} \check{\psi}_\alpha^{(0)}.$$

The zero-order wave functions have now the form of Eqs. (13), (15) with  $u_\uparrow = v_\downarrow \equiv u$  and  $u_\downarrow = v_\uparrow \equiv v$  where

$$u = \frac{1}{\sqrt{2}} \left[ 1 + \frac{v_F p}{E_0} \right]^{1/2}, \quad v = \frac{1}{\sqrt{2}} \left[ 1 - \frac{v_F p}{E_0} \right]^{1/2}. \quad (37)$$

The expansion coefficients satisfy

$$C_{\alpha\gamma} [E_\alpha - E_\gamma^{(0)}] = \sum_\beta C_{\alpha\beta} H_{\gamma\beta} \quad (38)$$

where  $H_{\gamma\beta} \equiv \langle \check{\psi}_\gamma^{(0)+} | \check{H}^{(1)} | \check{\psi}_\beta^{(0)} \rangle$ .

Consider the state  $\alpha = 1$ . Since the difference  $E_1 - E_1^{(0)} = E_1 - E_3^{(0)} \equiv \delta E_1$  in Eq. (38) is small, the coefficients  $C_{12}$  and  $C_{14}$  are proportional to the perturbation, while  $C_{11}$  and  $C_{13}$  are of the order unity. The coefficients  $C_{11}^{(0)}, C_{13}^{(0)}$  in the leading approximation satisfy the secular equations (B5), (B6) (see Appendix B) which yield  $\delta E_{1,3} = \mp \tilde{E}_1$ ,

$$\tilde{E}_1 = \sqrt{\left( \frac{\mu v_F p}{E_0} - E_D \right)^2 + \frac{v_F^2 [\mathbf{p} \times \mathbf{k}]^2 |\Delta|^2}{4p^2 E_0^2}} \quad (39)$$

and

$$C_{11}^{(0)} = C_{33}^{(0)} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\mu v_F p}{\tilde{E}_1} - \frac{E_D}{\tilde{E}_1}}, \quad (40)$$

$$C_{13}^{(0)} = C_{31}^{(0)} = \frac{i \text{sign}([\mathbf{p} \times \mathbf{k}]\mathbf{z})}{\sqrt{2}} \sqrt{1 - \frac{\mu v_F p}{\tilde{E}_1} - \frac{E_D}{\tilde{E}_1}}. \quad (41)$$

The coefficients obey the normalization  $|C_{11}^{(0)}|^2 + |C_{13}^{(0)}|^2 = |C_{33}^{(0)}|^2 + |C_{31}^{(0)}|^2 = 1$ . In the same way we find  $\delta E_{2,4} = \pm \tilde{E}_2$ . The energy correction  $\tilde{E}_2$  and the coefficients  $C_{22}^{(0)} = C_{44}^{(0)}$  and  $C_{24}^{(0)} = C_{42}^{(0)}$  are obtained from  $\tilde{E}_1$ ,  $C_{11}^{(0)}$ , and  $C_{13}^{(0)}$ , respectively, by replacing  $\mathbf{p} \rightarrow -\mathbf{p}$ .

For  $k/p \ll \mu/|\Delta|$  we have

$$\tilde{E}_{1,2} = \frac{\mu v_F p}{E_0} \mp E_D$$

and  $C_{11} = C_{33} = C_{22} = C_{44} = 1$ ,  $C_{13} = C_{31} = C_{24} = C_{42} = 0$  which agrees with the result of the linear approximation in  $k$ . The coefficients  $C_{ik}$  taken for  $v_F k \ll \mu$  coincide with Eqs. (24)-(25) in the limit  $\mu \ll E$ . In the limit  $\mu = 0$  we have

$$\tilde{E}_1 = \tilde{E}_2 \equiv \tilde{E} = (v_F p / E_0) \sqrt{E_D^2 + k^2 |\Delta|^2 / 4 p^2}$$

which agrees with Eq. (21).

Now we insert all four solutions into the expression Eq. (5) for the current. Removing the divergence by subtracting the current in the normal state we find

$$\mathbf{j} = e v_F^2 \mathbf{k} \int_0^\infty \frac{p dp}{2\pi} \left[ - \left( 1 + \frac{|\Delta|^2}{E_0^2} \right) \frac{d}{dE_0} \tanh \frac{E_0}{2T} + \frac{1}{v_F p} - \frac{v_F^2 p^2}{E_0^3} \tanh \frac{E_0}{2T} \right]. \quad (42)$$

It does not depend on  $\mu$ .

Consider low temperatures,  $T \ll |\Delta|$ . The first term vanishes and we obtain

$$\mathbf{j} = e v_F^2 \mathbf{k} \int_0^\infty \frac{p dp}{2\pi} \left[ \frac{1}{v_F p} - \frac{v_F^2 p^2}{E_0^3} \right] = \frac{e |\Delta| \mathbf{k}}{\pi} \quad (43)$$

which is the same result as in the linear approximation, Eq. (34).

#### IV. DISCUSSION AND COMPARISON

As we already mentioned in the Introduction, the present paper studies the model<sup>7,12</sup> that assumes Cooper pairing between electrons (holes) belonging to the same sublattice in the configurational space. This is evident, in fact, from the self-consistency equation Eq. (3), which contains the scalar product of the spinors  $\hat{u}$  and  $\hat{v}$ . However, other scenarios of the superconducting pairing are possible, as well. In particular, one can use the approach which is based on the Landau Fermi-liquid theory

which operates with the quasiparticles corresponding to the eigenstates of the normal-state Hamiltonian [spinors  $\hat{a}_\uparrow$  and  $\hat{a}_\downarrow$  in Eq. (11)]. All essential properties are then derived based on the quasiparticle energy spectrum. The Fermi-liquid approach (with some variations) was used in Refs. 6,8–11. The dilemma of “intra-sublattice interaction only” vs. “interaction between true quasiparticles of the normal graphene” was also discussed in connection with other collective modes in graphene<sup>19</sup>. In general, this dilemma can be resolved only on the basis of the detailed microscopic analysis of the particular interaction mechanism.

Nevertheless, the main outcome of our analysis is that the superconducting behaviors calculated within the two aforementioned approaches are qualitatively very similar, though intra-sublattice interaction requires a more involved algebra (4-fold matrices rather than 2-fold matrices in the Fermi-liquid approach). Quantitatively, however, Eq. (32) is slightly different from the result of Ref. 10. In particular, the current in the limit  $\mu \gg |\Delta|$ , Eq. (33), is twice as large as in Ref. 10. This factor 2 appears simply because the model with two times smaller number of the degrees of freedom (only one valley with a Dirac cone in the Brillouin zone) has been considered in the cited paper. For large  $\mu$ , there should otherwise be no difference in the superconducting properties, since the both models practically coincide with that for usual superconductors. However, the low- $\mu$  limit differs already by factor 4. The latter is obviously a manifestation of a more subtle difference between these two models.

Apart from this numerical difference, the global features of the superconducting graphene are insensitive to the choice of the pairing model. The main message of Ref. 10 is confirmed that the supercurrent and the superconducting electron density are finite at any doping level for all temperatures below the critical temperature; in particular, they do not disappear in the limit  $\mu = 0$  contrary to the claim of Ref. 7. We have in fact shown that the low-doping limit  $\mu \rightarrow 0$ , being degenerate in the excitation energies, is not any special in the sense of the supercurrent: The supercurrent obtained within the linear approximation in the gradient of the order parameter phase,  $k \ll |\Delta|/v_F$ , is the same irrespectively of the relation between  $v_F k$  and  $\mu$ . The crucial difference between the superconducting graphene and the usual BCS superconductor is that the supercurrent density in the low-doping limit at  $T = 0$  is proportional to the order parameter  $\Delta$  rather than to the total electron density.

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### Appendix A: Current in the linear approximation

We start with Eq. (29). Using Eqs. (13), (15), and (24), (25) we find after averaging over momentum direc-

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$$\mathbf{j} = -ev_F \sum_{\mathbf{p}} \frac{\mathbf{p}}{p} \left[ \tanh \frac{E_{\uparrow} + E_D}{2T} - \tanh \frac{E_{\uparrow} - E_D}{2T} + \tanh \frac{E_{\downarrow} + E_D}{2T} - \tanh \frac{E_{\downarrow} - E_D}{2T} \right] \\ - ev_F^2 \mathbf{k} \sum_{\mathbf{p}} \left[ \frac{|u_{\uparrow}^* u_{\downarrow} + v_{\uparrow}^* v_{\downarrow}|^2}{E_{\uparrow} - E_{\downarrow}} \left( \tanh \frac{E_{\uparrow}}{2T} - \tanh \frac{E_{\downarrow}}{2T} \right) + \frac{|v_{\uparrow}^* u_{\downarrow} - u_{\uparrow}^* v_{\downarrow}|^2}{E_{\uparrow} + E_{\downarrow}} \left( \tanh \frac{E_{\uparrow}}{2T} + \tanh \frac{E_{\downarrow}}{2T} \right) \right]. \quad (\text{A1})$$


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This expression formally diverges for large  $v_F p$  due to contribution from regions far from the Dirac points. As we discussed already in Section III we remove this spurious divergence by transforming the zero-order terms. With Eqs. (16), (17) we have in the zero-order approximation

$$\mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p}) = -\mathbf{j}_{-\mathbf{K}}^{(0)}(\mathbf{p}) \\ = -ev_F \left[ \frac{v_F p - \mu}{E_{\uparrow}} \tanh \frac{E_{\uparrow}}{2T} + \frac{v_F p + \mu}{E_{\downarrow}} \tanh \frac{E_{\downarrow}}{2T} \right] \frac{\mathbf{p}}{p}. \quad (\text{A2})$$

The contribution from the zero-order term has the form of Eq. (30) of surface integral over a remote sphere in the momentum space. Using Eq. (A2) we find

$$\mathbf{j}^{(0)} = - \int \frac{d\phi}{(2\pi)^2} [(\mathbf{p} \cdot \mathbf{k}) \mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p})]_{p \rightarrow \infty} \\ = ev_F^2 \mathbf{k} \int_0^\infty \frac{p dp}{2\pi} \left[ \frac{1}{v_F p} \right]. \quad (\text{A3})$$

When added to Eq. (A1), this compensates the diverging terms there. As a result, we obtain the converging expression, Eq. (31). The same result can be obtained if we subtract the normal current, i.e., Eq. (A1) for  $\Delta = 0$ .

For  $T \ll |\Delta|$  the terms  $\cosh^{-2}(E_{\uparrow, \downarrow}/2T)$  are small while  $\tanh(E_{\uparrow, \downarrow}/2T) = 1$ . Therefore, the first line in Eq. (31) vanishes. The current becomes

$$\mathbf{j} = ev_F^2 \mathbf{k} \int_0^\infty \frac{p dp}{2\pi} \left[ \frac{1}{v_F p} - \frac{2|v_{\uparrow}^* u_{\downarrow} - u_{\uparrow}^* v_{\downarrow}|^2}{E_{\uparrow} + E_{\downarrow}} \right]. \quad (\text{A4})$$

Calculating the current with help of Eqs. (16), (17) we obtain Eq. (32).

tions in the second line

### Appendix B: Current in the degenerate case

#### 1. Wave functions for weak doping $\mu \ll \Delta$

Consider Eq. (38) for the state  $\alpha = 1$ . We have within the linear approximation in  $H^{(1)}$

$$C_{11} [E_1 - E_1^{(0)}] = \sum_{\beta} C_{1\beta} H_{1\beta}, \quad (\text{B1})$$

$$C_{13} [E_1 - E_3^{(0)}] = \sum_{\beta} C_{1\beta} H_{3\beta}, \quad (\text{B2})$$

and

$$C_{12} [E_1 - E_2^{(0)}] = \sum_{\beta} C_{1\beta} H_{2\beta}, \quad (\text{B3})$$

$$C_{14} [E_1 - E_4^{(0)}] = \sum_{\beta} C_{1\beta} H_{4\beta}. \quad (\text{B4})$$

Since

$$E_1 - E_1^{(0)} = E_1 - E_3^{(0)} \equiv \delta E_1$$

are small, in Eqs. (B1) and (B2) we can take the coefficients in the zero order approximation. As a result  $C_{12}$  and  $C_{14}$  are small (i.e., are proportional to the perturbation), while  $C_{11}$  and  $C_{13}$  are of the order unity. We put  $C_{11} = C_{11}^{(0)}$ ,  $C_{13} = C_{13}^{(0)}$  while  $C_{12} = C_{12}^{(1)}$ ,  $C_{14} = C_{14}^{(1)}$ , and find up to the first order terms

$$C_{11}^{(0)} \delta E_1^{(1)} = C_{11}^{(0)} H_{11} + C_{13}^{(0)} H_{13}, \quad (\text{B5})$$

$$C_{13}^{(0)} \delta E_1^{(1)} = C_{11}^{(0)} H_{31} + C_{13}^{(0)} H_{33}, \quad (\text{B6})$$

while

$$2E_0 C_{12}^{(1)} = C_{11}^{(0)} H_{21} + C_{13}^{(0)} H_{23}, \quad (\text{B7})$$

$$2E_0 C_{14}^{(1)} = C_{11}^{(0)} H_{41} + C_{13}^{(0)} H_{43}. \quad (\text{B8})$$

The similar equations are obtained for the other state  $\alpha = 3$  which belongs to the same energy  $E_0$ .

We have

$$H_{\alpha\beta} = \frac{v_F}{2} \left( \hat{u}_\alpha^{+(0)} \hat{\sigma} \cdot \mathbf{k} \hat{u}_\beta^{(0)} + \hat{v}_\alpha^{+(0)} \hat{\sigma} \cdot \mathbf{k} \hat{v}_\beta^{(0)} \right) - \mu \left( \hat{u}_\alpha^{+(0)} \hat{u}_\beta^{(0)} - \hat{v}_\alpha^{+(0)} \hat{v}_\beta^{(0)} \right).$$

Therefore

$$H_{11} = -H_{33} = \frac{v_F \mathbf{p} \cdot \mathbf{k}}{2p} - \frac{\mu v_F p}{E_0},$$

$$H_{13} = -H_{31} = \frac{i v_F ([\mathbf{z}_0 \times \mathbf{p}] \mathbf{k})}{2p} \frac{|\Delta|}{E_0},$$

and

$$H_{21} = H_{21} = H_{43} = H_{34} = -\mu \frac{|\Delta|}{E_0}, \quad (\text{B9})$$

$$H_{23} = -H_{32} = -i \frac{v_F p}{E_0} \frac{v_F ([\mathbf{p} \times \mathbf{k}] \mathbf{z})}{2p}, \quad (\text{B10})$$

$$H_{41} = -H_{14} = -i \frac{v_F p}{E_0} \frac{v_F ([\mathbf{p} \times \mathbf{k}] \mathbf{z})}{2p}. \quad (\text{B11})$$

Secular equations (B5), (B6) determine  $\delta E_{1,3}$  together with the coefficients  $C_{11}^{(0)}$ ,  $C_{13}^{(0)}$ . The first-order corrections to  $C_{11}$  and  $C_{13}$  are found from Eqs. (B1) and (B2) written up to the second-order terms. Using Eqs. (B9), (B10), and (B11) we obtain  $C_{11}^{(1)} = C_{13}^{(1)} = 0$ . The coefficients  $C_{12}^{(1)}$  and  $C_{14}^{(1)}$  are determined by Eqs. (B7) and (B8). In the same way we find  $C_{31}^{(1)} = C_{33}^{(1)} = 0$  together with the coefficients  $C_{32}^{(1)}$  and  $C_{34}^{(1)}$ . Calculations for the states  $\alpha = 2, 4$  can be done in exactly the same way.

Using the obtained coefficients we can rewrite Eq. (5) for the current in the form

$$\mathbf{j} = -e v_F \sum_{\mathbf{p}} \left[ A_1^{(0)} \left( \tanh \frac{E_0 - \tilde{E}_1}{2T} - \tanh \frac{E_0 + \tilde{E}_1}{2T} \right) - A_2^{(0)} \left( \tanh \frac{E_0 - \tilde{E}_2}{2T} - \tanh \frac{E_0 + \tilde{E}_2}{2T} \right) + 2 \left( A_1^{(1)} - A_2^{(1)} \right) \tanh \frac{E_0}{2T} \right], \quad (\text{B12})$$

where

$$A_1^{(0)} = (|C_{11}^{(0)}|^2 - |C_{13}^{(0)}|^2) \hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\uparrow + 2uv \left( C_{11}^{(0)*} C_{13}^{(0)} - C_{13}^{(0)*} C_{11}^{(0)} \right) \hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\downarrow,$$

$$A_2^{(0)} = (|C_{22}^{(0)}|^2 - |C_{24}^{(0)}|^2) \hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\uparrow + 2uv \left( C_{22}^{(0)*} C_{24}^{(0)} - C_{24}^{(0)*} C_{22}^{(0)} \right) \hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\downarrow,$$

and

$$A_1^{(1)} = (u^2 - v^2) E_0^{-1} H_{23} \hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\downarrow,$$

$$A_2^{(1)} = (u^2 - v^2) E_0^{-1} H_{14} \hat{a}_\uparrow^\dagger \hat{\sigma} \hat{a}_\downarrow.$$

The current in Eq. (B12) takes the form

$$\mathbf{j} = 2e \sum_{\mathbf{p}} \left[ \frac{\partial \tilde{E}_1}{\partial \mathbf{k}} \left( \tanh \frac{E_0 - \tilde{E}_1}{2T} - \tanh \frac{E_0 + \tilde{E}_1}{2T} \right) + \frac{\partial \tilde{E}_2}{\partial \mathbf{k}} \left( \tanh \frac{E_0 - \tilde{E}_2}{2T} - \tanh \frac{E_0 + \tilde{E}_2}{2T} \right) - \frac{v_F^2 p^2}{E_0^3} \frac{v_F^2 [\mathbf{p} \times \mathbf{k}] \times \mathbf{p}}{2} \tanh \frac{E_0}{2T} \right]. \quad (\text{B13})$$

For  $v_F k \ll \mu$  the this equation coincides with Eq. (A1) taken for  $\mu \ll \Delta$ . Expanding it in small  $\tilde{E}_1$  and  $\tilde{E}_2$  we find

$$\mathbf{j} = -2e \sum_{\mathbf{p}} \left[ \frac{\partial}{\partial \mathbf{k}} \left( \tilde{E}_1^2 + \tilde{E}_2^2 \right) \frac{d}{dE_0} \tanh \frac{E_0}{2T} + \frac{v_F^2 p^2}{E_0^3} \frac{v_F^2 [\mathbf{p} \times \mathbf{k}] \times \mathbf{p}}{p^2} \tanh \frac{E_0}{2T} \right]$$

which yields after integration over the momentum angle

$$\mathbf{j} = -e v_F^2 \mathbf{k} \sum_{\mathbf{p}} \left[ \left( 1 + \frac{|\Delta|^2}{E_0^2} \right) \frac{d}{dE_0} \tanh \frac{E_0}{2T} + \frac{v_F^2 p^2}{E_0^3} \tanh \frac{E_0}{2T} \right]. \quad (\text{B14})$$

Subtraction of the normal-state current returns us to Eq. (42).

## 2. Undoped graphene

The Bogoliubov–de Gennes equations can be solved exactly for the undoped case  $\mu = 0$  where the dispersion equation (20) becomes bi-quadratic with the energy spectrum Eq. (21). We shall write down the solution of the Bogoliubov–de Gennes equations for the vector  $\mathbf{k}$  parallel to the axis  $x$  ( $k = k_x$ ). Then

$$E_\pm^2 = (\sqrt{|\Delta|^2 + v_F^2 p_x^2} \pm v_F k/2)^2 + v_F^2 p_y^2, \quad (\text{B15})$$

and one may check by substitution that the four orthonormalized solutions of the Bogoliubov–de Gennes equations are given by the spinors

$$\hat{u} = \frac{1}{2} \sqrt{1 \pm \frac{p_x}{\sqrt{|\Delta|^2/v_F^2 + p_x^2}}} \begin{pmatrix} \pm \frac{E_\pm}{|E_\pm|} e^{-i\phi_\pm/2} \\ e^{i\phi_\pm/2} \end{pmatrix},$$

$$\hat{v} = \frac{1}{2} \sqrt{1 \mp \frac{p_x}{\sqrt{|\Delta|^2/v_F^2 + p_x^2}}} \begin{pmatrix} \pm e^{-i\phi_\pm/2} \\ \frac{E_\pm}{|E_\pm|} e^{i\phi_\pm/2} \end{pmatrix}, \quad (\text{B16})$$

where the phase factors are given by

$$e^{i\phi_\pm} = \frac{\sqrt{|\Delta|^2/v_F^2 + p_x^2} \pm k/2 \pm i p_y}{\sqrt{(\sqrt{|\Delta|^2/v_F^2 + p_x^2} \pm k/2)^2 + p_y^2}}.$$

Using Eq. (B16) one finds the terms which determine the  $x$  component of the supercurrent:



$$\begin{aligned}\hat{u}^\dagger \hat{\sigma}_x \hat{u} &= \pm \frac{1}{2} \frac{E_\pm}{|E_\pm|} \sqrt{1 \pm \frac{p_x}{\sqrt{|\Delta|^2/v_F^2 + p_x^2}}} \cos \phi_\pm \approx \pm \frac{1}{2} \frac{E_\pm}{|E_\pm|} \sqrt{1 \pm \frac{p_x}{\sqrt{|\Delta|^2/v_F^2 + p_x^2}}} \left( 1 \pm \frac{kp_y^2}{(|\Delta|^2/v_F^2 + p^2)^{3/2}} \right), \\ \hat{v}^\dagger \hat{\sigma}_x \hat{v} &= \pm \frac{1}{2} \frac{E_\pm}{|E_\pm|} \sqrt{1 \pm \frac{p_x}{\sqrt{|\Delta|^2/v_F^2 + p_x^2}}} \cos \phi_\pm \approx \pm \frac{1}{2} \frac{E_\pm}{|E_\pm|} \sqrt{1 \mp \frac{p_x}{\sqrt{|\Delta|^2/v_F^2 + p_x^2}}} \left( 1 \pm \frac{kp_y^2}{(|\Delta|^2/v_F^2 + p^2)^{3/2}} \right),\end{aligned}$$

where finally we keep only terms linear in  $k$ . Collecting all the terms in the expression Eq. (5) we arrive at

Eq. (B14).

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